

Hitting Time Distribution for finite states Markov Chain¹

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Abstract

Consider a Markov chain with finite state $\{0, 1, \dots, d\}$ and absorbing at state d . We give out the generation functions (or Laplace transforms) for the absorbing time when the chain starts from any state i . The results generalize the well-known theorems for the birth-death (Karlin and McGregor [7], 1959) and the skip-free (Brown and Shao [1], 1987) Markov chain starts from state 0. Our proof is directly and simple.

Keywords: Markov chain, absorbing time, generation functions, Laplace transforms, eigenvalues.

Mathematics Subject Classification (2010): 60E10, 60J10, 60J27, 60J35.

1 Introduction

For the birth-death and the skip-free (upward jumps may be only of unit size, and there is no restriction on downward jumps) Markov chain with finite state $\{0, 1, \dots, d\}$ and absorbing at state d , an well-known interesting property for the absorbing time is that it is distributed as a summation of d independent geometric (or exponential) random variables.

There are many authors give out different proofs to the results. For the birth and death chain, the well-known results can be traced back to Karlin and McGregor ([7], 1959) Keilson ([8], 1971; [9]). Kent and Longford ([10], 1983) proved the result for the discrete time version (nearest random walk) although they have not specified the result as usual form (section 2, [10]). Fill ([4], 2009) gave the first stochastic proof to both nearest random walk and birth and death chain cases via duality which was established in [2]. Diaconis and Miclo ([3], 2009) presented another probabilistic proof for birth and death chain. Very recently, Gong, Mao and Zhang ([6], 2012) gave a similar result in the case that the state space is \mathbb{Z}^+ . For the skip-free chain, Brown and Shao ([1], 1987) first proved the result in continuous time situation; Fill ([5], 2009) gave a stochastic proof to both discrete and continuous time cases also by using the duality, and considered the general finite-state Markov chain situation when the chain starts from the state 0.

The purpose of this paper is to consider a general Markov chain with finite-state $\{0, 1, \dots, d\}$ and absorbing at state d . We give out the generation functions (or Laplace transforms) for the

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absorbing time when the chain starts from any state i (not just from state 0 only). In particular, the results generalize the well-known theorems for the birth-death (Karlin and McGregor [7], 1959) and the skip-free (Brown and Shao [1], 1987) Markov chain.

Our proof is to calculate directly the generation functions (or Laplace transforms) of the absorbing times by iterating method, which have been used for the skip-free Markov chain by Zhou ([11]). After revised the method in [11], we found the proof is very simple and enable us to deal with the general finite-state Markov chain starts from any state i . The key idea is to consider directly the absorbing time $\tau_{i,d}$ starting from any state i .

2 Discrete time

Define the transition probability matrix P as

$$P = \begin{pmatrix} r_0 & p_{0,1} & p_{0,2} & \cdots & p_{0,d-1} & p_{0,d} \\ q_{1,0} & r_1 & p_{1,2} & \cdots & p_{1,d-1} & p_{1,d} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ q_{d-1,0} & q_{d-1,1} & q_{d-1,2} & \cdots & r_{d-1} & p_{d-1,d} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{(d+1) \times (d+1)}, \quad (2.1)$$

and for $1 \leq j \leq d+1$, we denote $A_j(s)$ as the $d \times d$ sub-matrix by deleting the $(d+1)^{th}$ line and the j^{th} row of the matrix $I_{d+1} - sP$.

Let $\tau_{i,d}$ be the absorbing time of state d starting from i and $f_i(s)$ be the generation function of $\tau_{i,d}$,

$$f_i(s) = \mathbb{E}s^{\tau_{i,d}} \quad \text{for } 0 \leq i \leq d-1. \quad (2.2)$$

Theorem 2.1. For $0 \leq i \leq d-1$,

$$f_i(s) = (-1)^{d-i} \frac{\det A_{i+1}(s)}{\det A_{d+1}(s)}. \quad (2.3)$$

Proof. By decomposing the first step, the generation function of $\tau_{i,d}$ satisfy, for $0 \leq i \leq d-1$,

$$\begin{aligned} f_i(s) &= r_i s f_i(s) + p_{i,i+1} s f_{i+1}(s) + p_{i,i+2} s f_{i+2}(s) + \cdots + p_{i,d-1} s f_{d-1}(s) + p_{i,d} s \\ &\quad q_{i,i-1} s f_{i-1}(s) + q_{i,i-2} s f_{i-2}(s) + \cdots + q_{i,0} s f_0(s). \end{aligned} \quad (2.4)$$

These system of d equations are linear about $f_0(s), f_1(s), \dots, f_{d-1}(s)$. Use Cramer's Rule, we can solve from these system of d equations and get (2.3). \square

Immediately, we obtain the results for the skip-free discrete time Markov chain (Fill [5], 2009).

Corollary 2.1. If we assume for $j-i > 1$, $p_{i,j} = 0$. We have

$$f_0(s) = \prod_{i=0}^{d-1} \left[\frac{(1 - \lambda_i)s}{1 - \lambda_i s} \right], \quad (2.5)$$

where $\lambda_0, \dots, \lambda_{d-1}$ are the d non-unit eigenvalues of P .

In particular, if all of the eigenvalues are real and nonnegative, then the hitting time is distributed as the sum of d independent geometric random variables with parameters $1 - \lambda_i$.

Proof. Note that, 1 is evidently an eigenvalue of P . So on the one hand $\det(I_{d+1} - sP) = (1 - s) \prod_{i=0}^{d-1} (1 - \lambda_i s)$ (where $\lambda_0, \dots, \lambda_{d-1}$ are the d non-unit eigenvalues of P); on the other hand from (2.1) we have $\det(I_{d+1} - sP) = (1 - s) \times \det A_{d+1}(s)$; as a consequence we get

$$\det A_{d+1}(s) = \prod_{i=0}^{d-1} (1 - \lambda_i s). \quad (2.6)$$

From (2.1) and the definition of A_j , it is easy to get

$$\det A_1(s) = (-1)^d p_{0,1} p_{1,2} \cdots p_{d-1,d} s^d. \quad (2.7)$$

By (2.3) and (2.6) we have

$$\det A_1(1) = (-1)^d f_0(1) \cdot \det A_{d+1}(1) = (-1)^d f_0(1) \cdot \prod_{i=0}^{d-1} (1 - \lambda_i).$$

On the other hand, from (2.7) we get $\det A_1(1) = (-1)^d p_{0,1} p_{1,2} \cdots p_{d-1,d}$ and recall that $f_0(1) = 1$ by (2.2), we obtain that

$$p_{0,1} p_{1,2} \cdots p_{d-1,d} = \prod_{i=0}^{d-1} (1 - \lambda_i). \quad (2.8)$$

Then by (2.7) and (2.8)

$$\det A_1(s) = (-1)^d \prod_{i=0}^{d-1} (1 - \lambda_i) s^d, \quad (2.9)$$

and (2.5) is immediate from (2.6) and (2.9). \square

3 Continuous time

Define the generator Q as

$$Q = \begin{pmatrix} -\gamma_0 & \alpha_{0,1} & \alpha_{0,2} & \cdots & \alpha_{0,d-1} & \alpha_{0,d} \\ \beta_{1,0} & -\gamma_1 & \alpha_{1,2} & \cdots & \alpha_{1,d-1} & \alpha_{1,d} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \beta_{d-1,0} & \beta_{d-1,1} & \beta_{d-1,2} & \cdots & -\gamma_{d-1} & \alpha_{d-1,d} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{(d+1) \times (d+1)},$$

and for $1 \leq j \leq d+1$, we denote $\tilde{A}_j(s)$ as the $d \times d$ sub-matrix by deleting the $(d+1)^{th}$ line and the j^{th} row of the matrix $I_{d+1} - sP$. Let $\tau_{i,d}$ be the absorbing time of state d starting from i and $\tilde{f}_i(s)$ be the Laplace transform of $\tau_{i,d}$.

$$\tilde{f}_i(s) = \mathbb{E} e^{-s\tau_{i,d}}.$$

It is well known that the chain on the finite state has an simple structure. The process starts at i , it stay there with an Exponential (γ_i) time, then jumps to $i+j$ with probability $\frac{\alpha_{i,i+j}}{\gamma_i}$, to $i-k$ with probability $\frac{\beta_{i,i-k}}{\gamma_i}$.

Theorem 3.1.

$$\tilde{f}_i(s) = (-1)^{d-i} \frac{\det \tilde{A}_{i+1}}{\det \tilde{A}_{d+1}}, \quad \text{for } 0 \leq i \leq d-1, \quad (3.1)$$

Proof. By decomposing the trajectory at the first jump, $0 \leq i \leq d-1$,

$$\begin{aligned} \tilde{f}_i(s) &= \frac{\gamma_i}{\gamma_i + s} \frac{\alpha_{i,i+1}}{\gamma_i} \tilde{f}_{i+1}(s) + \frac{\gamma_i}{\gamma_i + s} \frac{\alpha_{i,i+2}}{\gamma_i} \tilde{f}_{i+1}(s) + \cdots + \frac{\gamma_i}{\gamma_i + s} \frac{\alpha_{i,d-1}}{\gamma_i} \tilde{f}_{d-1}(s) + \frac{\gamma_i}{\gamma_i + s} \frac{\alpha_{i,d}}{\gamma_i} + \\ &\quad + \frac{\gamma_i}{\gamma_i + s} \frac{\beta_{i,i-1}}{\gamma_i} \tilde{f}_{i-1}(s) + \frac{\gamma_i}{\gamma_i + s} \frac{\beta_{i,i-2}}{\gamma_i} \tilde{f}_{i-2}(s) + \cdots + \frac{\gamma_i}{\gamma_i + s} \frac{\beta_{i,0}}{\gamma_i} \tilde{f}_0(s) \\ &= \frac{\alpha_{i,i+1}}{\gamma_i + s} \tilde{f}_{i+1}(s) + \cdots + \frac{\alpha_{i,d-1}}{\gamma_i + s} \tilde{f}_{d-1}(s) + \frac{\alpha_{i,d}}{\gamma_i + s} + \frac{\beta_{i,i-1}}{\gamma_i + s} \tilde{f}_{i-1}(s) + \cdots + \frac{\beta_{i,0}}{\gamma_i + s} \tilde{f}_0(s) \end{aligned}$$

These system of d equations are linear about $\tilde{f}_0(s), \tilde{f}_1(s), \dots, \tilde{f}_{d-1}(s)$. Use Cramer's Rule, we can solve from these system of d equations and get (3.1). \square

Immediately, we obtain the results for the skip-free continuous time Markov chain (Brown and Shao [1], 1987).

Corollary 3.1. *If we assume for $j - i > 1$, $\alpha_{i,j} = 0$. We have*

$$\varphi_d(s) = \prod_{i=0}^{d-1} \frac{\lambda_i}{s - \lambda_i},$$

where λ_i are the d non-zero eigenvalues of $-Q$.

In particular, if all of the eigenvalues are real and nonnegative, then the hitting time is distributed as the sum of d independent exponential random variables with parameters λ_i .

Proof. The proof is similar as Corollary 2.1, we can calculate that $\det \tilde{A}_{d+1} = \prod_{i=0}^{d-1} (s - \lambda_i)$, $\det \tilde{A}_1 = (-1)^d \alpha_{0,1} \alpha_{1,2} \cdots \alpha_{d-1,d} = (-1)^d \prod_{i=0}^{d-1} \lambda_i$. \square

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